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World-line techniques for resumming gluon radiative corrections at the cross-section level

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Abstract. We employ the Polyakov world-line path-integral version of QCD to identify and resum at leading perturbative order enhanced radiative gluon contributions to the Drell–Yan type ($q\bar{q}$ pair annihilation) cross-sections. We emphasize that this is the first time that world-line techniques are applied to cross-section calculations.

1 Introduction

The path, rather than the functional, integral casting of a relativistic quantum system has a long history, going back to Fock [1], Feynman [2], and Schwinger [3]. In the course of time it has received substantial contributions from several authors (see, e.g., [4–9] to name just a few). However, it was not realized until recent developments in string theory in the context of effective actions (see [10–12], and [13] for a recent review) that first-quantization methodologies in high energy theory can compete with second-quantization ones. Of particular interest to us here is the Polyakov world-line path integral [14], which employs world-line paths weighted by a spin factor with the aim to describe the propagation of particle-like entities in Euclidean space-time. In fact, Polyakov's intention was to use this construction as a simple prototype for discussing string quantization. Hence, for his purposes it was sufficient to consider the simple case of a free, spin-1/2 particle-like entity. Motivated by this, two of the present authors [15] explored the possibility of transcribing the matter, spin-1/2, field sector of a gauge theory into a Polyakov world-line path-integral form. In these works, it was established, for both Abelian and non-Abelian gauge systems, that this is, indeed, possible with the spin factor making explicitly its entrance in the resulting expression, while the dynamics enters through a Wilson line (loop) factor defined on each given path.

Due to the Gaussian character (with respect to the Dirac fields) of the fermionic sector of physically relevant gauge-field theories, the aforementioned transcription into a Polyakov world-line path integral refers to the full system. This means that one's way of thinking should be readjusted to the idea that the second-quantization formalism, associated with the field theoretical mode of description, can be replaced by a new, but equivalent, structure that is based on space-time path integrals. The quantities of central importance defined on these paths are then the spin factor and the Wilson line (loop), the latter becoming an indigenous element of the theory, as it enters at the level of its definition.

In a number of papers – see, for instance, [6, 15–17] – we have employed the path-integral casting of either QED or QCD, to study infra-red (IR) factorization and the ensuing behavior of Green's functions and amplitudes in a resummed perturbative context at the two-, three- and four-point function level. Roughly speaking, the aforementioned isolation of the long-distance physics in these theories emerges through the ability to identify a special set of space-time paths having a very simple geometrical profile which is shared, in a restricted (but directly relevant to the physics of the process) neighborhood, by each and every contour entering the path integral. In a Euclidean space-time context, the single (multiplicative) renormalization constant, carried by this special family of paths, automatically factorizes out [18] their contribution to amplitudes/cross-sections, given that it also accompanies the rest of the paths. The more complex geometrical structure of the latter, simply implicates additional ultraviolet (UV) singularities which can be absorbed into conventional wave-function and coupling-constant renormalizations. This clean, geometrically based, argument, which singularly underlines the world-line description, will be further elucidated through the main exposition in the sections to follow. Minkowski space subtleties, associated with the light cone, which are encountered in the particular processes under study, will require separate attention.

Perhaps the most important accomplishment of this paper is that it extends world-line techniques to cross-section calculations for the first time. To be sure, the situ-

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ation presently considered refers to the hypothetical situation where the scattering process involves quarks and associated gluon radiation without reference to hadrons. It does, nevertheless, fall within the spirit that marks our approach to IR issues in QCD [16,17]: Once off-mass shell IR protection is employed – by an amount that exceeds $\Lambda_{\rm QCD}$ – one actually tests how far one can go by remaining strictly within the confines of QCD before attempting to make contact with real hadrons. Granted the opposite route, from hadrons to quarks and gluons, via the use of quantities like structure/fragmentation functions, the employment of tools such as the operator product expansion, etc., constitutes a more realistic procedure for investigating the same physical problems. On the other hand, an effort which bases its considerations on a fundamental theoretical framework in order to arrive at "cross-sections" does present merits and interests of its own, an example of which will be presented below. In this context, the philosophy underlying our approach to the IR domain of QCD is closer in spirit to the one articulated by Ciafaloni in [19], the only difference being that we shall keep a more pragmatic (and less ambitious) course by focusing our attention on cross-section expressions.

Letting these comments suffice for an introductory exposition, we now proceed to display the organization of the paper, which is as follows. In the next section, we exhibit the world-line expression for the full fermionic Green's function and subsequently employ it to construct corresponding expressions for DY-type QCD amplitudes/cross-sections. Section 3 furnishes, with the aid of the appendix, our basic calculations associated with one virtual-gluon exchanges for the specified special set of trajectories. The resulting expression explicitly reveals the threshold enhancement factor, whereas the task of virtual-gluon resummation is performed, via the aid of the renormalization group, in Sect. 4. Section 5 deals with the resummation of contributions from real-gluon emission. Finally, in the last section, we further discuss our results and present our conclusions.

2 Basic world-line expressions for amplitudes and cross-sections

Consider the full two-point (fermionic) Green's function in the presence of an external gluonic field. The Polyakov path-integral expression, in Euclidean space-time,

$$\begin{aligned}
&\mathrm{i}G_{ij}(x,y|A)\\
&=\int_0^\infty \mathrm{d}T\mathrm{e}^{-Tm^2}\int_{\substack{x(0)=x\\x(T)=y}}\mathcal{D}x(t)\left[m-\frac{1}{2}\gamma\cdot\dot{x}(T)\right]\\
&\times P\exp\left(\frac{\mathrm{i}}{4}\int_0^T \mathrm{d}t\sigma_{\mu\nu}\omega_{\mu\nu}\right) \qquad (1)\\
&\times\exp\left[-\frac{1}{4}\int_0^T\mathrm{d}t\dot{x}^2(t)\right]P\exp\left[\mathrm{i}g\int_0^T\mathrm{d}t\dot{x}\cdot A(x(t))\right]_{ij},
\end{aligned}$$

displays the basic world-line features pertaining to this quantity. Here, and below, P denotes the usual path ordering of the integrals. The first thing to point out is that a given path of the matter-field quantum, starting at x and ending at y between respective "proper-time" values 0 and T, also enters a Wilson line factor. The latter, being the sole carrier of the dynamics, separates itself from the rest of the factors in the path integral which are associated with geometrical properties of paths traversed by spin-1/2 particle entities. The most notable such quantity is the so-called spin factor [14], $P\exp\left[(\mathrm{i}/4)\int_0^T\mathrm{d}t\sigma\cdot\omega\right]$, where $\omega_{\mu\nu}=(T/2)(\ddot{x}_\mu\dot{x}_\nu-\dot{x}_\mu\ddot{x}_\nu)$, accounting, in a geometrical way, for the spin-1/2 nature of the propagating particle. Accordingly, our perturbative expansions should be perceived of in terms of (Euclidean) space-time paths involving a "proper time" parameter and *not* in terms of Feynman diagrams. As it turns out [20], in the perturbative context, the structure of matter particle contours, entering the path integral, is determined by the points where a momentum change takes place, i.e., points where a gauge-field line (real or virtual) attaches itself on the (fermionic) matter-field path. The almost everywhere non-differentiability of these contours is residing precisely at these points. A major effort, in this paper, will be devoted to the extension of the world-line formalism to expressions for cross-sections corresponding to the particular processes of $q\bar{q}$ annihilation.

From the world-line point of view, the process we intend to study involves fermionic matter particle (quark) paths that commence at x and end at y, being forced to pass through an intermediate point z, where a momentum transfer Q takes place. This means that the Green's (vertex-type) function we shall be dealing with has the following form (Γ_μ denotes some Clifford–Dirac algebra element)

$$\begin{aligned}
&V_{\mu,ij}(y,z,x|A)=G_{ik}(y,z|A)\Gamma_\mu G_{kj}(z,x|A) \qquad (2)\\
&=\int_0^\infty\mathrm{d}T\mathrm{e}^{-Tm^2}\int_0^T\mathrm{d}s\int_{\substack{x(0)=x\\x(T)=y}}\mathcal{D}x(t)\delta\left(x(s)-z\right)\mathcal{G}_\mu\left(\dot{x},s\right)\\
&\times\exp\left[-\frac{1}{4}\int_0^T\mathrm{d}t\dot{x}^2(t)\right]P\exp\left[\mathrm{i}g\int_0^T\mathrm{d}t\dot{x}(t)\cdot A(x(t))\right]_{ij},
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{G}_\mu\left(\dot{x},s\right)\equiv&\left[m-\frac{1}{2}\gamma\cdot\dot{x}(T)\right]P\exp\left(\frac{\mathrm{i}}{4}\int_s^T\mathrm{d}t\sigma\cdot\omega\right)\\
&\times\Gamma_\mu\left[m-\frac{1}{2}\gamma\cdot\dot{x}(s)\right]\\
&\times P\exp\left(\frac{\mathrm{i}}{4}\int_0^s\mathrm{d}t\sigma\cdot\omega\right). \qquad (3)
\end{aligned}$$

It is especially important to realize that in our approach off-shellness is naturally parameterized in terms of the finite size of the matter particle contours and realistically accounts for the fact that quarks reside inside a hadron (m can be viewed as an effective quark mass).

Going over to momentum space, we write

$$\tilde{V}_{\mu,ij}(p,p'|z|A)$$

$$
\begin{aligned}
&= \int_0^\infty \mathrm{d}T \mathrm{e}^{-Tm^2} \int_0^T \mathrm{d}s \int \mathcal{D}x(t) \delta\left(x(s)-z\right) \mathcal{G}_\mu\left(\dot{x}, s\right) \\
&\times \exp\left[-\frac{1}{4}\int_0^T \mathrm{d}t \dot{x}^2(t) + \mathrm{i}p\cdot x(0) + \mathrm{i}p'\cdot x(T)\right] \\
&\times P\exp\left[\mathrm{i}g \int_0^T \mathrm{d}t \dot{x}(t)\cdot A(x(t))\right]_{ij} \\
&\equiv \sum_{C^z} \tilde{\Gamma}_\mu[C^z] P\exp\left[\mathrm{i}g\int_{C^z} \mathrm{d}x\cdot A(x)\right]_{ij}, \qquad (4)
\end{aligned}
$$

where C^z denotes a generic path forced to pass through point z, at which the momentum Q is imparted.

For a process of the type $q+\bar{q} \to$ lepton pair + X the "amplitude" expression reads

$$
\begin{aligned}
\Delta_{\mu,ij} &= \bar{v}(p',s')(-\mathrm{i}\gamma\cdot p' + m)\tilde{V}_{\mu,ij}(\mathrm{i}\gamma\cdot p + m)u(p,s) \\
&\equiv \sum_{C^z} \tilde{I}_{\mu,p'p}[C^z] P\exp\left[\mathrm{i}g\int_{C^z} \mathrm{d}x\cdot A(x)\right]_{ij}, \qquad (5)
\end{aligned}
$$

with the second, comprehensive, expression to be understood having recourse to (4).

For the cross-section, we need to employ the following quantity, which we implicitly display in Minkowski spacetime after straightforward adjustments:

$$
\begin{aligned}
\Delta^\dagger_\mu \Delta_\nu &= \sum_{\bar{C}^{z'}} \sum_{C^z} \tilde{I}^\dagger_{\mu,p'p}[\bar{C}^{z'}] \tilde{I}_{\nu,p'p}[C^z] \\
&\times \mathrm{Tr}\left\{ \bar{P}\exp\left[\mathrm{i}g\int_{\bar{C}^{z'}} \bar{dx}^\rho A_\rho(\bar{x})\right] \right. \\
&\times \left. P\exp\left[-\mathrm{i}g\int_{C^z} \mathrm{d}x^\sigma A_\sigma(x)\right] \right\}, \qquad (6)
\end{aligned}
$$

where $\bar{P}$ denotes anti-path ordering. Even though not explicitly displayed, the cross-section acquires a path-integral form, which has the following characteristics.

(1) Paths C^z and $\bar{C}^{z'}$ are forced to pass through points z and z', respectively, where the momentum transfer occurs (see Fig. 1). The distance $b \equiv |z-z'|$ serves as a measure of how far apart the two conjugate contours can venture away from each other and will be referred to as the impact parameter.
(2) The traversal of $\bar{C}^{z'}$ is made in the opposite sense relative to C^z. If we now let the two paths join at one end by using translational invariance, while we allow the other two ends of the contour to close at infinity, then we obtain the formation of a Wilson loop.
(3) Under these circumstances, the Wilson loop formation guarantees the gauge invariance of the expression for the cross-section.

On the other hand, by keeping the contour lengths finite, but very large, thereby placing the quarks off-mass-shell, gauge invariance will still continue to hold to the order of approximation we employ in our computations, given that the off-mass-shellness serves at the same time as an IR cutoff.

Up to this point our considerations have been centered around the geometrical profile of the paths entering the world-line casting of QCD, the main conclusion being that, for the process considered, the relevant contours entering the path integral are marked by a characteristic point, where a momentum transfer is imparted and that they are *open* for the amplitude and *closed* (or almost so) for the cross-section. Armed with this information, we now turn our attention to the Wilson factor which contains all the dynamics of the given process. The obvious task in front of us is to assess its implications once the gauge fields are quantized, i.e., once the Wilson factor is inserted into a *functional* integral weighted by the exponential of the Yang–Mills action. We display the quantity of interest as follows

$$
\begin{aligned}
\mathcal{W} &= \left\langle \mathrm{Tr}\left\{ \bar{P}\exp\left[\mathrm{i}g\int_{\bar{C}^{z'}} \bar{\mathrm{d}x}^\mu A_\mu(\bar{x})\right]\right\}_{\mathrm{A}} \right. \\
&\times \left. \left\{ P\exp\left[-\mathrm{i}g\int_{C^z} \mathrm{d}x^\nu A_\nu(x)\right]\right\}_{\mathrm{A}} \right\rangle \\
&\equiv \langle \mathrm{Tr}(U^\dagger(\bar{C}^{z'})U(C^z))\rangle. \qquad (7)
\end{aligned}
$$

In the above expression, $\{\cdots\}_{\mathrm{A}}$ signifies the expectation value with respect to the gauge-field functional integral which, in this work, will be considered in the context of perturbation theory. Note in the same context that a virtual gluon attaching itself with both ends to the fermionic world-line, entering the amplitude, corresponds to a correlator between a pair of gauge fields originating from the expansion of the Wilson factor. On the other hand, for an emitted "real" gluon from the fermionic line, the correlator is between an "external" and a Wilson line gauge field[1]. The overall situation is depicted in Fig. 2. At the cross-section level, now, "real" gluons are integrated with respect to "propagators" linking together the two conjugate contours, while their polarization vectors are summed over (cut propagators). This is precisely what $\langle\cdots\rangle$ signi-

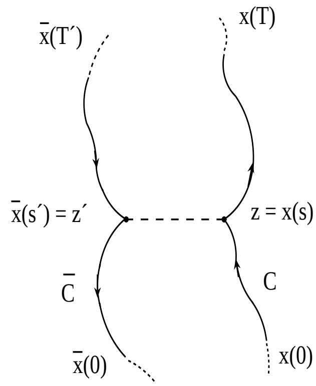

Fig. 1. Illustration of two conjugate contours C and $\bar{C}$ entering the world-line path integral, "talking" to each other at points z and z', where the momentum transfer for the physical process takes place. The distance $|z-z'|$ is referred to as the impact parameter

[1] One will, of course, also encounter correlators that involve gauge fields from the non-linear terms of the Yang–Mills action. These, however, do not enter the leading logarithmic considerations relevant for our considerations

fies in the last equation, as it brackets both Wilson line factors.

This marks a crucial difference to conventional approaches (for example, [21–23]), wherein the Drell–Yan process is discussed in a context where IR factorization is based on the eikonal approximation for soft amplitudes. Wilson loop expectation values, entering this scheme, are evaluated along contours corresponding to classical trajectories – along with a segment which lies on the light cone, introduced in order to secure gauge invariance. In our case, by contrast, Wilson contours are built in at a foundational level, being themselves an integral part of the description of the full QCD. Accordingly, factorization properties for us are integrally connected with the renormalization properties of Wilson loops studied in the more general context of [18]. In the light of the above remarks, let us proceed to display the first-order (in perturbation theory) expression for $\mathcal{W}$, which receives contributions from virtual gluons, viz., those attached at both ends of either the world-line contour C^z or $\bar{C}^{z'}$, as well as from "real" gluons linking these contours to each other (cf. Fig. 2). This expression reads

$$
\begin{aligned}
\mathcal{W}^{(2)} = {} & \mathrm{Tr}I - g^2 C_\mathrm{F} \int_0^T \mathrm{d}t_1 \int_0^T \mathrm{d}t_2 \theta\left(t_2 - t_1\right) \\
& \times \dot{x}^\mu\left(t_2\right) \dot{x}^\nu\left(t_1\right) D_{\mu\nu}\left(x(t_2) - x(t_1)\right) \\
& - g^2 C_\mathrm{F} \int_0^{T'} \mathrm{d}t_1' \int_0^{T'} \mathrm{d}t_2' \theta\left(t_1' - t_2'\right) \dot{\bar{x}}^\mu\left(t_2'\right) \dot{\bar{x}}^\nu\left(t_1'\right) \\
& \times \bar{D}_{\mu\nu}\left(\bar{x}(t_2') - \bar{x}(t_1')\right) \\
& - g^2 C_\mathrm{F} \int_0^T \mathrm{d}t \int_0^{T'} \mathrm{d}t' \dot{x}(t) \cdot \dot{\bar{x}}\left(t'\right) D_\mathrm{cut}\left(x(t) - \bar{x}(t)\right) \\
& + \mathcal{O}\left(g^4\right).
\end{aligned} \tag{8}
$$

It becomes obvious from their structure that the first two non-trivial terms correspond to virtual-gluon contributions – one per conjugate branch –, while the third one is associated with "real" gluon emission. Finally, concerning the gluon propagators entering the above equation, we shall be employing their Feynman-gauge form without loss of generality due to gauge invariance. In particular we have, in D dimensions,

$$
\begin{aligned}
D_{\mu\nu}(x) &= -\mathrm{i} g_{\mu\nu} \mu^{4-D} \int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{\mathrm{e}^{-\mathrm{i}k\cdot x}}{k^2 + \mathrm{i}0_+} \\
&= g_{\mu\nu} \frac{1}{4\pi^2} \left(-\pi\mu^2\right)^{(4-D)/2} \frac{\Gamma(D/2-1)}{(x^2 - \mathrm{i}0_+)^{(D/2)-1}},
\end{aligned} \tag{9}
$$

whereas

$$
\begin{aligned}
D_\mathrm{cut}(x) &= \mu^{4-D} \int \frac{\mathrm{d}^D q}{(2\pi)^D} 2\pi\delta(q^2)\theta(q^0)\mathrm{e}^{-\mathrm{i}q\cdot x} \\
&= \frac{1}{4\pi^2} \left(-\pi\mu^2\right)^{(4-D)/2} \frac{\Gamma(D/2-1)}{\left[(x_0^2 - \mathrm{i}0_+)^2 - \boldsymbol{x}^2\right]^{(D/2)-1}}.
\end{aligned} \tag{10}
$$

From here on and for the sake of notational simplicity, we shall simply write $D(x)$ instead of $D_\mathrm{cut}(x)$.

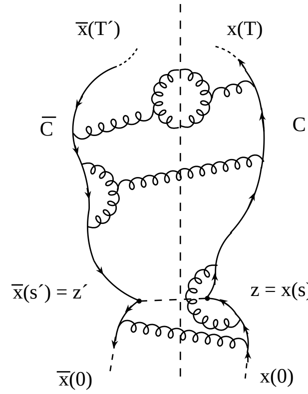

Fig. 2. Virtual gluon radiative corrections of various sorts and "real" gluon lines with their ends attached on each of the two depicted contours at the cross-section level

As already established by other methods, the perturbative expansion (8) is plagued by large threshold logarithms leading to the need for factorization and resummation. This is precisely the task we are about to undertake within our framework.

3 First-order virtual-gluon corrections in the vicinities of points z and z'

The space-time mode of description of the Polyakov world-line formalism puts us into the position to promote the following argument: The point z (or z'), where the momentum transfer Q is imparted, marks the presence of a neighborhood around it, no matter how infinitesimal in size this might be, whose geometrical structure is shared by *all* fermionic paths entering the path integral. Specifically, there will be a derailment (cusp formation), whose opening angle will be fixed unambiguously, since it is determined by the momentum transfer. It follows that the contributions to the amplitude and cross-section from the immediate vicinities of each of the two cusps is a common feature of all contours and eventually factorizes. In this section we shall determine the first-order perturbative term corresponding exactly to this factor.

Consider now the neighborhood of point z on the contour C^z. Expanding around this point, we write

$$
x^\mu(t) = x^\mu(s) + (t-s)\dot{x}^\mu(s \pm 0) + \ldots \tag{11}
$$

with $v^\mu = \dot{x}^\mu(s-0)$ and $v'^\mu = \dot{x}^\mu(s+0)$ being entrance and exit four-velocities, respectively, with respect to z.

Adjusting our notation by re-parameterizing the contour so that the zero value is assigned to point z, the relevant quantity to compute, to first perturbative order, becomes

$$
\begin{aligned}
U_{C,\mathrm{S}}^{(2)} = {} & 1 \\
& - g^2 C_\mathrm{F} \left[\int_{-\sigma}^0 \mathrm{d}t_1 \int_{-\sigma}^0 \mathrm{d}t_2 \theta\left(t_2 - t_1\right) v^\mu v^\nu D_{\mu\nu}\left(vt_2 - vt_1\right) \right. \\
& + \int_0^\sigma \mathrm{d}t_1 \int_0^\sigma \mathrm{d}t_2 \theta\left(t_2 - t_1\right) + v'^\mu v'^\nu D_{\mu\nu}\left(v't_2 - v't_1\right)
\end{aligned} \tag{12}
$$

$$+\int_0^{-\sigma} \mathrm{d}t_1 \int_0^{\sigma} \mathrm{d}t_2 \theta\left(t_2-t_1\right) v'^{\mu} v^{\nu} D_{\mu\nu}\left(v' t_2 - v t_1\right)\Big] .$$

It is clear that the above expression corresponds to the first term in (8), which monitors a virtual-gluon exchange occurring on contour C.

From the above considerations it follows that the main contribution to each double integral comes from the common limit $t_1, t_2 \to 0$. Suppose now, the other limit is to be determined by demanding that its contribution to the integrals is of vanishing importance. Then, such a requirement automatically isolates those contours, whose only significant geometrical characteristic is that the four-velocities to approach and depart from point z are fixed, denoted by v^{μ} and v'^{μ} respectively. The same, of course, happens for point z', but in the reverse order. This justifies the subscript S in $U_{C,\mathrm{S}}$, which stands for "smooth". Let us also observe that the omitted terms in (12) will contain negative powers of σ, whose dimension in the denominator is $(\text{mass})^2$. Neglecting their presence means that σ should be very large in magnitude and hence it should be related to an IR cutoff, i.e., $\sigma \simeq \lambda^{-2}, \lambda > \Lambda_{\mathrm{QCD}}$.

In Euclidean space-time, now, *every path* will share the geometrical structure we are focusing on in some neighborhood of the point z or (z'), no matter how close to these points one has to come. At the same time, the UV singularities, exhibited by this restricted set of paths, will entail expressions that solely depend on the two four-velocities and the opening angle. Paths of more complex geometrical structure, on the other hand, will certainly exhibit these UV singularities *plus* additional ones[2]. It follows that – in Euclidean space-time at least – the restricted set of trajectories, by exclusively carrying the corresponding (multiplicative) renormalization constant, *factorizes* from the rest of the expression for the amplitude and/or cross-section. In a Minkowski space-time context, which will be considered next, we should anticipate the existence of additional contributions to (12), due to the light-cone structure that cannot be assigned to each and every contour and, therefore, cannot be factorized. Let us, then, go over to Minkowski space-time, where we have two distinct possibilities for defining an infinitesimally small neighborhood around z. The first one, to be labelled (a), reads

$$(x-x')^2 = \mathcal{O}(\epsilon^2), \quad \text{with} \quad v_\mu \simeq v'_\mu, \quad \text{for all} \quad \mu, \tag{13}$$

where ϵ $(\leq Q^{-1})$ is a small length scale. The second alternative, to be labelled (b), can be typically represented by

$$\begin{aligned}
&(x-x')^2 = \mathcal{O}(\epsilon^2) \quad \text{with} \quad |v-v'|^2 = \mathcal{O}(\lambda^2) \\
&\quad \text{but} \quad (v_+ - v'_+) \simeq \mathcal{O}(Q) \\
&\quad \text{and} \quad (v_- - v'_-) \simeq \mathcal{O}\left(\frac{\lambda^2}{Q}\right) \\
&\quad \Rightarrow (v_+ - v'_+)(v_- - v'_-) = \mathcal{O}(\lambda^2)
\end{aligned} \tag{14}$$

that is equivalently effected via the condition $v_+ \gg v'_+$, $v_- \simeq v'_-$. All in all, there are four different configurations: $+ \leftrightarrow -$ and prime $\leftrightarrow$ no-prime entering this case.

We denote case (a) as "uniformly soft", given that the considered gluon exchanges take place in a neighborhood whose smallness pertains to all directions. Case (b), on the other hand, will be referred to as "jet" since gluon emission occurs under circumstances, where entrance and exit four-velocities differ from each other significantly along one or the other of the light-cone directions. Particular implications stemming from this, purely Minkowskian, case as far as the factorization issue is concerned, will be considered later on.

Let us commence our calculations by taking up the first $\mathcal{O}(g^2)$ term entering the right hand side of (12). Since this only involves the branch of the contour C^z entering point z, we obtain the same expression regardless of whether or not a uniformly soft or a jet configuration is being considered. It reads

$$\begin{aligned}
I_1 &= \int_{-\sigma}^{0} \mathrm{d}t_1 \int_{-\sigma}^{0} \mathrm{d}t_2 \theta\left(t_2-t_1\right) v^{\mu} v^{\nu} D_{\mu\nu}\left(v t_2 - v t_1\right) \\
&= -\frac{1}{8\pi^2}\left(-\pi\mu^2 L_1^2\right)^{(4-D)/2} \\
&\quad \times \Gamma\left(\frac{D}{2}-1\right) \frac{1}{D-3} \frac{1}{2-D/2},
\end{aligned} \tag{15}$$

where[3] $L_1 = \sigma|v|$. The second term has the same structure as the first one (it involves the exiting branch of C^z) and therefore produces a similar result:

$$\begin{aligned}
I_2 = &-\frac{1}{8\pi^2}\left(-\pi\mu^2 L_2^2\right)^{(4-D)/2} \\
&\times \Gamma\left(\frac{D}{2}-1\right) \frac{1}{D-3} \frac{1}{2-D/2},
\end{aligned} \tag{16}$$

with $L_2 = \sigma|v'|$.

A couple of remarks are in order at this point. First, even though the length scales L_1 and L_2 are both large, being proportional to σ, they will be of the same order of magnitude for case (a), whereas for case (b), one scale will be negligible in comparison with the other. Accordingly, the total expression to the amplitude for the uniformly soft contribution will be twice as large as that of the jet-like one. This having been said, we shall denote the dominant length scale by L $(\simeq L_1$ and/or $L_2)$, when it enters our final expressions, and set it equal to $1/\lambda$, recognizing that it is of the same order as the IR cutoff. Second, in order to avoid the double counting resulting from the fact that each branch has been "cut-off" at distance L away from z, where gluon emission occurring at the endpoints will be offset by a similar one, but opposite in sign, from that portion of the contour that continues to stretch out to infinity, the final expressions for the end-point singularities should be multiplied by a factor of 1/2. Equivalently, one might think of this compensation as actually identifying the missing energy of the gluon emission at the extremities

[2] Actually, the standard UV singularities of perturbative field theories associated with β-functions, coupling-constant and wave-function renormalization, pertain to almost everywhere non-differentiable paths

[3] Note that v has the dimension of mass as our "time" parameter σ has the dimension of $(\text{mass})^{-2}$

of the path with the off-mass-shellness. In fact, this is what we have been implying all along when claiming that finite contours signify off-mass-shellness.

Turning our attention to the contribution resulting from a virtual-gluon exchange from the entrance to the exit branch, with respect to z, we consider the quantity

$$I_3 = \int_{-\sigma}^{0} \mathrm{d}t_1 \int_0^{\sigma} \mathrm{d}t_2 v'^{\mu} v^{\nu} D_{\mu\nu} \left(v' t_2 - v t_1\right)$$
$$= \frac{1}{4\pi^2} \left(-\pi\mu^2\right)^{(4-D)/2} \Gamma\left(\frac{D}{2} - 1\right) v \cdot v' \tag{17}$$
$$\times \int_0^{\sigma} \mathrm{d}t_1 \int_0^{\sigma} \mathrm{d}t_2 \left(t_1^2 v^2 + t_2^2 v'^2 + 2t_1 t_2 v \cdot v' - \mathrm{i}0_+\right)^{1-(D/2)} .$$

For case (a) it assumes the form (recall that $v \cdot v'$ is negative)

$$I_3^{(\mathrm{a})} = \frac{1}{4\pi^2} \left(-\pi \frac{\mu^2}{\lambda^2}\right)^{(4-D)/2} \Gamma\left(\frac{D}{2} - 1\right) \frac{v \cdot v'}{|v||v'|} \tag{18}$$
$$\times \int_0^1 \mathrm{d}t_1 \int_0^1 \mathrm{d}t_2 \left(t_1^2 + t_2^2 + 2t_1 t_2 \frac{v \cdot v'}{|v||v'|} - \mathrm{i}0_+\right)^{1-(D/2)} .$$

As shown in the appendix, one then determines (γ_{E} is Euler's constant)

$$I_3^{(\mathrm{a})} = \frac{1}{8\pi^2} \gamma \coth \gamma \frac{1}{2 - \frac{D}{2}} + \frac{1}{8\pi^2} \gamma \coth \gamma \ln\left(\frac{\mu^2}{\lambda^2} \pi \mathrm{e}^{2+\gamma_{\mathrm{E}}}\right), \tag{19}$$

where $\cosh \gamma = w = -v \cdot v' / (|v||v'|) \geq 1$.

In all of the above expressions, as well as in those that will follow, we have ignored:

(i) all imaginary terms that will drop out when contributions (for virtual gluons) from the conjugate contour are taken into account, and

(ii) finite, μ-independent terms that will cancel out when real-gluon contributions to the cross-section are included.

Collecting all terms, we deduce, for the "uniformly smooth" part,

$$I_1^{(\mathrm{a})} + I_2^{(\mathrm{a})} + I_3^{(\mathrm{a})} = \frac{1}{8\pi^2} (\gamma \coth \gamma - 1) \frac{1}{2 - \frac{D}{2}} \tag{20}$$
$$+ \frac{1}{8\pi^2} (\gamma \coth \gamma - 1) \ln\left(\frac{\mu^2}{\lambda^2} \pi \mathrm{e}^{2+\gamma_{\mathrm{E}}}\right).$$

Concerning the "jet" part of the computation, we only need to consider $I_3^{(\mathrm{b})}$ because[4] the expression for $I_1^{(\mathrm{b})} + I_2^{(\mathrm{b})}$ is simply one half of that of $I_1^{(\mathrm{a})} + I_2^{(\mathrm{a})}$. A typical term entering $I_3^{(\mathrm{b})}$ ($v_- \gg v'_+$) is

$$I_3^{(\mathrm{b})} = \frac{1}{4\pi^2} \left(-\pi\mu^2\right)^{(4-D)/2} \Gamma\left(\frac{D}{2} - 1\right) v \cdot v' \tag{21}$$
$$\times \int_0^{\sigma} \mathrm{d}t_1 \int_0^{\sigma} \mathrm{d}t_2 \left(t_1^2 v^2 + 2t_1 t_2 v \cdot v' - \mathrm{i}0_+\right)^{1-(D/2)} ,$$

whose computation suffices to furnish each of the other three terms as well.

It is shown in the latter part of the appendix that one obtains

$$I_3^{(\mathrm{b})} = \frac{1}{16\pi^2} \frac{1}{\left(2 - \frac{D}{2}\right)^2} + \frac{1}{16\pi^2} \frac{1}{2 - \frac{D}{2}} \ln\left(\frac{\mu^2}{\lambda^2} \pi \mathrm{e}^{\gamma_{\mathrm{E}}}\right)$$
$$+ \frac{1}{32\pi^2} \ln^2\left(\frac{\mu^2}{\lambda^2} \pi \mathrm{e}^{\gamma_{\mathrm{E}}}\right) + \text{const.} \tag{22}$$

It is duly observed that the singularity structure of the above expression is γ-independent. In fact, the "jet" configuration is a Minkowski space feature and is connected to "gluon" emission in the + or the − light-cone coordinates direction. This result is in accord with Wilson loop studies in Minkowski space, wherein the relevant contour lies partly on the light cone [22]. Subtracting the pole terms in the $\overline{\mathrm{MS}}$ scheme, we arrive at the finite part of the overall result. For the uniformly soft contribution, in particular, we get

$$(I_1^{(\mathrm{a})} + I_2^{(\mathrm{a})} + I_3^{(\mathrm{a})})_{\mathrm{fin}} = \frac{1}{8\pi^2} (\gamma \coth \gamma - 1) \ln\left(\frac{\mu^2}{\bar{\lambda}^2}\right), \tag{23}$$

while the jet contribution reads

$$(I_1^{(\mathrm{b})} + I_2^{(\mathrm{b})} + 4I_3^{(\mathrm{b})})_{\mathrm{fin}} = \frac{1}{8\pi^2} \ln^2\left(\frac{\mu^2}{\bar{\lambda}^2}\right) - \frac{1}{16\pi^2} \ln \frac{\mu^2}{\bar{\lambda}^2}, \tag{24}$$

where we have set $\bar{\lambda}^2 \equiv 4\lambda^2 \mathrm{e}^{-2\gamma_{\mathrm{E}}}$. The above relation takes into account all four different configurations contributing to $I_3^{(\mathrm{b})}$.

Gathering all terms, we arrive at the following overall result for the second-order contribution stemming from contour C^z:

$$U_{C,\mathrm{S}}^{(2)} = 1 - \frac{\alpha_{\mathrm{s}}}{2\pi} C_{\mathrm{F}} \left[(\gamma \coth \gamma - 1) \ln\left(\frac{\mu^2}{\bar{\lambda}^2}\right) - \frac{1}{2} \ln\left(\frac{\mu^2}{\bar{\lambda}^2}\right)\right.$$
$$\left. + \ln^2\left(\frac{\mu^2}{\bar{\lambda}^2}\right)\right]. \tag{25}$$

A similar result is obtained also for contour $\bar{C}^{z'}$.

Noting that $\gamma \coth \gamma = \ln\left(Q^2/m^2\right)$ (for $Q^2 \gg m^2$), with $Q^2 = (p + p')^2$, we recognize that the well-known perturbative enhancements occurring as $Q^2 \to \infty$ are associated with the eikonal-type trajectories upon which our present calculations have been based. One, now, realizes that these trajectories define *threshold* conditions, with respect to the given momentum exchange Q, for the process under consideration, since they leave no room for space-time contour fluctuations. In the following section, we shall treat the resummation of these enhanced contributions to leading logarithmic order. We shall, furthermore, identify a correction factor associated with those terms in (25) not involving the enhancement factor $\ln(Q^2/m^2)$.

[4] Recall the remark following (16)

4 Resummation of enhanced contributions from virtual gluons

The family of world-line paths to which the considerations in the previous section refer was used in order to deal with all (virtual) single-gluon exchanges, consistent with the simple geometrical configuration of two constant four-velocities making a fixed angle γ between them (in Euclidean formulation). Among these gluons there will be "hard" ones (upper limit Q) and "soft" ones (lower limit set by $\bar{\lambda}$). What is debited to the former and what to the latter group of gluons is, of course, relative. It is precisely the role of the renormalization scale μ, entering through the need to face UV divergences arising even for the restricted family of paths, to provide the dividing line. The corresponding renormalization-group equation reflects the fact that the scale μ is arbitrary and that physical results do not depend on it. A straightforward application of this fact will enable us to resum the enhanced, virtual-gluon contribution to the amplitude in leading logarithmic order, as well as to obtain a bona-fide correction term.

To bring the above discussion into a concrete form, let us first consider a separation, good to order $1/Q^2$, of the cusp contribution, which can be factorized from the amplitude U_C, entering (7), on the basis of what has been determined so far. Hence, we write

$$U_C = U_{C,\mathrm{cusp}}\left(\frac{Q^2}{m^2}, \frac{\mu^2}{\bar{\lambda}^2}\right)\hat{U}_C\left(\frac{Q^2}{\mu^2}, \frac{\mu^2}{\bar{\lambda}^2}\right) + \mathcal{O}\left(\frac{1}{Q^2}\right), \tag{26}$$

with

$$U^{(2)}_{C,\mathrm{cusp}} = 1 - \frac{\alpha_{\mathrm{s}}}{2\pi}C_{\mathrm{F}}\left(\gamma\coth\gamma - 1\right)\ln\left(\frac{\mu^2}{\bar{\lambda}^2}\right), \tag{27}$$

where we have normalized $U_{C,\mathrm{cusp}}$ to unity for $\gamma \to 0$. The designation "cusp", above, refers to that factor of the soft sector, which recognizes the angle γ. The factor $\hat{U}_C$, on the other hand, includes *both*: (i) soft contributions – related to the dependence on the quantity $\mu^2/\bar{\lambda}^2$ – *and* (ii) hard ones – depending on the quantity Q^2/μ^2.

It is convenient to take the logarithmic derivative of (26) with respect to Q^2:

$$\frac{\mathrm{d}}{\mathrm{d}\ln Q^2}\ln U_C = \frac{\mathrm{d}}{\mathrm{d}\ln Q^2}\ln U_{C,\mathrm{cusp}} + \frac{\mathrm{d}}{\mathrm{d}\ln Q^2}\ln \hat{U}_C + \mathcal{O}\left(\frac{1}{Q^2}\right). \tag{28}$$

The μ-independence of physical results leads to the renormalization-group equation whose ultimate justification has to do with the multiplicative renormalization of the soft (cusp-angle dependent) factor. Indeed, the latter is detached from collinear emission and totally complies with the Euclidean space-time properties of Wilson loops for which the results of [18] fully apply. Specifically, we write

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu}\frac{\mathrm{d}}{\mathrm{d}\ln Q^2}\ln\hat{U}_C = -\frac{\mathrm{d}}{\mathrm{d}\ln\mu}\frac{\mathrm{d}}{\mathrm{d}\ln Q^2}\ln U_{C,\mathrm{cusp}} = \Gamma_{\mathrm{cusp}}(\alpha_{\mathrm{s}}), \tag{29}$$

with Γ_{cusp} to be read off from (20)–(22) and (12):

$$\Gamma_{\mathrm{cusp}}(\alpha_{\mathrm{s}}) = \frac{\alpha_{\mathrm{s}}}{\pi}C_{\mathrm{F}} + \mathcal{O}(\alpha_{\mathrm{s}}^2). \tag{30}$$

From the second leg of (29), one obtains

$$\frac{\mathrm{d}}{\mathrm{d}\ln Q^2}\ln U_{C,\mathrm{cusp}} = -\int_{\bar{\lambda}^2}^{\mu^2}\frac{\mathrm{d}t}{2t}\Gamma_{\mathrm{cusp}}\left[\alpha_{\mathrm{s}}(t)\right], \tag{31}$$

which, in turn, gives

$$\frac{\mathrm{d}}{\mathrm{d}\ln Q^2}\ln\hat{U}_C = -\int_{\mu^2}^{Q^2}\frac{\mathrm{d}t}{2t}\Gamma_{\mathrm{cusp}}\left[\alpha_{\mathrm{s}}(t)\right] + \Gamma\left[\alpha_{\mathrm{s}}(Q^2)\right], \tag{32}$$

where we have defined

$$\frac{1}{2}\Gamma\left[\alpha_{\mathrm{s}}(Q^2)\right] \equiv \frac{\mathrm{d}}{\mathrm{d}\ln Q^2}\ln\hat{U}_C\left(\frac{Q^2}{\mu^2}, \frac{\mu^2}{\bar{\lambda}^2}\right)_{\mu^2=Q^2}. \tag{33}$$

Combining the last three equations, we have

$$\frac{\mathrm{d}}{\mathrm{d}\ln Q^2}\ln U_C = -\int_{\bar{\lambda}^2}^{Q^2}\frac{\mathrm{d}t}{2t}\Gamma_{\mathrm{cusp}}\left[\alpha_{\mathrm{s}}(t)\right] + \frac{1}{2}\Gamma\left[\alpha_{\mathrm{s}}(Q^2)\right], \tag{34}$$

where, in terms of $\ln U_C$, we write

$$\frac{1}{2}\Gamma\left[\alpha_{\mathrm{s}}(Q^2)\right] \equiv \frac{\mathrm{d}}{\mathrm{d}\ln Q^2}\ln U_C|_{\bar{\lambda}^2=Q^2}. \tag{35}$$

Setting $\mu^2 = Q^2$ in (26), we are led to the identification

$$\frac{1}{2}\Gamma\left[\alpha_{\mathrm{s}}(Q^2)\right] = \frac{\mathrm{d}}{\mathrm{d}\ln Q^2}\ln U_{C,\mathrm{cusp}}\left(\frac{Q^2}{m^2}, \frac{Q^2}{\bar{\lambda}^2}\right)\bigg|_{\bar{\lambda}^2=Q^2} + \frac{\mathrm{d}}{\mathrm{d}\ln Q^2}\ln\hat{U}_C\left(1, \frac{Q^2}{\bar{\lambda}^2}\right)\bigg|_{\bar{\lambda}^2=Q^2}. \tag{36}$$

To second order we have

$$\Gamma^{(2)}\left[\alpha_{\mathrm{s}}(Q^2)\right] = \frac{3}{2}\frac{C_{\mathrm{F}}}{\pi}\alpha_{\mathrm{s}}(Q^2) + \mathcal{O}(\alpha_{\mathrm{s}}^2). \tag{37}$$

Gathering our findings, we obtain our final, resummed result corresponding to the contour C. It reads

$$U_C = \exp\left\{-\int_{\bar{\lambda}^2}^{Q^2}\frac{\mathrm{d}t}{2t}\left[\ln\frac{Q^2}{t}\Gamma_{\mathrm{cusp}}(\alpha_{\mathrm{s}}(t)) - \Gamma(\alpha_{\mathrm{s}}(t))\right]\right\} \times U_{C,0}(\alpha_{\mathrm{s}}(Q^2)). \tag{38}$$

One notes that the second ("correction") term in the square brackets is associated with collinear emission (cf. (36)). Finally, the factor $U_{C,0}(\alpha_{\mathrm{s}}(Q^2))$ represents input from initial conditions *at the QCD level.* Clearly, the conjugate-contour term $U^{\dagger}(\bar{C}^{z'})$ can be treated in a completely analogous fashion.

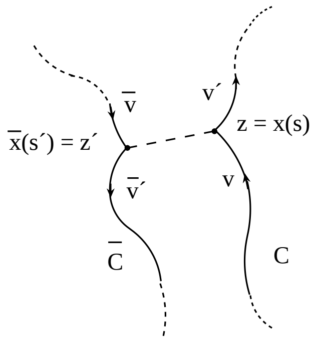

Fig. 3. Neighborhoods of respective points on two conjugate contours, where the momentum transfer takes place, and associated four-velocities

5 Resummation of enhanced contributions from real-gluon emission

We shall now turn our attention to real gluons and attempt to factorize cross-section contributions from neighborhoods around points z and z'. Note that this time we have to deal with gluons which connect two "opposite" neighborhoods while crossing the unitarity line (this situation is depicted in Fig. 3). The relevant scale promptly entering our considerations is the impact parameter $b = z-z'$, which must be eventually integrated over in order to get the physically measurable cross-section. Naturally, the short-distance cutoff in this integration will be provided by the (length) scale $1/|Q|$.

For the eikonal-type family of paths, and in first-order perturbation theory, the relevant quantity on which our quantitative considerations are to be based, i.e., the counterpart of (12), is given by

$$\begin{aligned} U^{(2)}_{C\bar{C},\mathrm{S}} = 1 + g^2 C_\mathrm{F} \Bigg[& \int_{-\sigma}^{0} \mathrm{d}t_1 \int_{-\sigma}^{0} \mathrm{d}t_2 v\cdot\bar{v} D\left(t_1 v - t_2\bar{v} + b\right) \\ & + \int_{0}^{\sigma} \mathrm{d}t_1 \int_{0}^{\sigma} \mathrm{d}t_2 v'\cdot\bar{v}' D\left(t_1 v' - t_2\bar{v}' + b\right) \\ & + \int_{-\sigma}^{0} \mathrm{d}t_1 \int_{0}^{\sigma} \mathrm{d}t_2 v\cdot\bar{v}' D\left(t_1 v - t_2\bar{v}' + b\right) \\ & + \int_{0}^{\sigma} \mathrm{d}t_1 \int_{-\sigma}^{0} \mathrm{d}t_2 v'\cdot\bar{v} D\left(t_1 v' - t_2\bar{v} + b\right) \Bigg], \end{aligned} \tag{39}$$

where the bar denotes four-velocities for the conjugate contour and the subscript cut is henceforth omitted.

To identify the leading behavior of $U^{(2)}_{C\bar{C},\mathrm{S}}$, with respect to b, we shall consider first the situation corresponding to $b = 0$. The subsequent emergence of UV divergences, once handled through dimensional regularization, will introduce a mass scale μ' that will be bounded from below by an IR cutoff λ and from above by the (mass) scale $1/b$. The resulting renormalization-group equation will facilitate the resummation of the leading terms, just as in the virtual-gluon case.

Let us start with our quantitative considerations by looking at the term

$$J_1(b) \equiv v\cdot\bar{v} \int_{-\sigma}^{0} \mathrm{d}t_1 \int_{-\sigma}^{0} \mathrm{d}t_2 D\left(t_1 v - t_2\bar{v} + b\right), \tag{40}$$

with $v^2 = \bar{v}^2 = -v\cdot\bar{v}$ (see Fig. 3).

Setting $b = 0$ and using the expression for the cut propagator as given by (10), we obtain

$$\begin{aligned} J_1(0) = & -\frac{1}{4\pi^2}\left(-\pi\mu'^2 L_1^2\right)^{(2-D/2)} \\ & \times \Gamma\left(\frac{D}{2}-1\right)\frac{1}{D-3}\frac{1}{4-D}\left[1-\left(2^{4-D}-1\right)\right], \end{aligned} \tag{41}$$

which actually coincides with what one would obtain if the regular propagator was substituted. The significance of this occurrence is that it leads to the same anomalous dimensions for the running of the real-gluon contribution to the cross-section as for the virtual part. This fact can be immediately verified via a direct comparison with (15).

Isolating the finite part of the above expression, we write

$$J_1^{(\mathrm{a}),\mathrm{fin}} = -\frac{1}{8\pi^2}\ln\left(\frac{\mu'^2}{\lambda^2}\right). \tag{42}$$

It is trivial to see that the same result holds also for $J_2^{(\mathrm{a}),\mathrm{fin}}$.

We next turn our attention to the term

$$J_3(b) \equiv v\cdot\bar{v}' \int_{-\sigma}^{0} \mathrm{d}t_1 \int_{0}^{\sigma} \mathrm{d}t_2 D\left(t_1 v - t_2\bar{v}' + b\right). \tag{43}$$

Its computation will concurrently allow us to determine $J_4(b)$, which corresponds to the exchange prime $\leftrightarrow$ noprime in the expression above.

Dimensionally regularizing the cut propagator, we then obtain

$$\begin{aligned} J_3(0) = & \frac{1}{4\pi^2}(-\pi\mu'^2)^{(4-D)/2}\Gamma\left(\frac{D}{2}-1\right) v\cdot\bar{v}' \\ & \times \int_0^\sigma \mathrm{d}t_1 \int_0^\sigma \mathrm{d}t_2 \left(t_1^2 v^2 + t_2^2\bar{v}'^2 + 2v\cdot\bar{v}' t_1 t_2 - \mathrm{i}0_+\right)^{1-D/2}. \end{aligned} \tag{44}$$

Once again we record, by referring to (17), coincidence of the singularities and, by extension, of associated anomalous dimensions between virtual- and real-gluon expressions that contribute to the cross-section.

For the "uniformly soft" configuration the corresponding result is

$$\begin{aligned} J_3^{(\mathrm{a})}(0) = & \frac{1}{4\pi^2}\left(-\pi\frac{\mu'^2}{\lambda^2}\right)^{(4-D)/2} \\ & \times \Gamma\left(\frac{D}{2}-1\right)\frac{v\cdot\bar{v}'}{|v||\bar{v}'|} \\ & \times \int_0^1 \mathrm{d}t_1 \int_0^1 \mathrm{d}t_2 \left(t_1^2 + t_2^2 + 2t_1 t_2\frac{v\cdot\bar{v}'}{|v||\bar{v}'|} - \mathrm{i}0_+\right)^{1-D/2}. \end{aligned} \tag{45}$$

Taking into consideration that

$$\frac{v\cdot\bar{v}'}{|v||\bar{v}'|} = \frac{v'\cdot\bar{v}}{|v'||\bar{v}|} = \cosh\gamma > 0,$$

we obtain

$$J_3^{(\mathrm{a})}(0) = J_4^{(\mathrm{a})}(0) = \frac{1}{4\pi^2}\left(-\pi\frac{\mu'^2}{\lambda^2}\right)^{(4-D)/2} \times \Gamma\left(\frac{D}{2}-1\right)\cosh\gamma \times \int_0^1 \mathrm{d}t_1 \int_0^1 \mathrm{d}t_2 \left(t_1^2+t_2^2+2t_1t_2\cosh\gamma - \mathrm{i}0_+\right)^{1-D/2}, \quad (46)$$

whose finite part reads

$$J_3^{(\mathrm{a}),\mathrm{fin}}(0) = J_4^{(\mathrm{a}),\mathrm{fin}}(0) = \frac{1}{8\pi^2}\gamma\coth\gamma\ln\left(\frac{\mu'^2}{\lambda^2}\right). \quad (47)$$

Turning now our attention to the "jet" configuration, we can actually go directly to $J_3^{(\mathrm{b})}(0)$, since $J_1^{(\mathrm{b})}(0) + J_2^{(\mathrm{b})}(0)$ furnishes half of the contribution of its uniformly soft counterpart, the reason being the same as the one given in the virtual-gluon case. We thus have

$$J_3^{(\mathrm{b})}(0) = \frac{1}{4\pi^2}\left(-\pi\mu'^2\right)^{(4-D)/2}\Gamma\left(\frac{D}{2}-1\right)\frac{v\cdot\bar{v}'}{|v|} \times \int_0^1 \mathrm{d}t_1\int_0^1 \mathrm{d}t_2\left(t_1^2+2t_1t_2\frac{v\cdot\bar{v}'}{|v|}-\mathrm{i}0_+\right)^{1-D/2}, \quad (48)$$

with an analogous expression holding also for $J_4^{(\mathrm{b})}(0)$.

For the finite parts of the "jet" contribution, one obtains

$$J_1^{(\mathrm{b}),\mathrm{fin}}(0)+J_2^{(\mathrm{b}),\mathrm{fin}}(0)+4\left[J_3^{(\mathrm{b}),\mathrm{fin}}(0)+J_4^{(\mathrm{b}),\mathrm{fin}}(0)\right] = \frac{1}{4\pi^2}\ln^2\left(\frac{\mu'^2}{\lambda^2}\right)-\frac{1}{8\pi^2}\ln\left(\frac{\mu'^2}{\lambda^2}\right). \quad (49)$$

Collecting our findings from the real-gluon analysis to the second-order level, we write for the finite contribution to the cross-section

$$U^{(2)}_{C\bar{C},\mathrm{S}} = 1+\frac{\alpha_\mathrm{s}}{\pi}C_\mathrm{F}\left[(\gamma\coth\gamma-1)\ln\left(\frac{\mu'^2}{\lambda^2}\right) - \frac{1}{2}\ln\left(\frac{\mu'^2}{\lambda^2}\right)+\ln^2\left(\frac{\mu'^2}{\lambda^2}\right)\right], \quad (50)$$

At the same time, the singularity structure of the full expression for the cross-section entails a multiplicative renormalization factor, which is common to all "Wilson loop" configurations entering its description, but which is the *only* one that pertains to the family of eikonal-type paths under consideration. The reasoning is, of course, identical to the one given for the virtual-gluon case. Therefore, the corresponding contribution to the cross-section factorizes and the same resummation procedure can be employed as for the virtual-gluon case. As already observed, the anomalous dimension is in both cases the same. There are, however, the following notable differences. First, the upper limit for the momentum of real-gluon emission is $1/b^2$ instead of Q^2. Second, there is a difference of sign, which becomes evident by comparing (25) with (50). Finally, no hard real-gluon emission enters our considerations – by definition. In this light, it is practically self-evident that the resummed expression for real-gluon emission becomes

$$U_{C\bar{C}} = \exp\left\{\int_{\bar{\lambda}^2}^{c/b^2}\frac{\mathrm{d}t}{t} \times \left[\ln\frac{Q^2}{t}\Gamma_\mathrm{cusp}\left(\alpha_\mathrm{s}(t)\right)-\Gamma\left(\alpha_\mathrm{s}(t)\right)\right]\right\}U_{C\bar{C},0}, \quad (51)$$

where $c = 4\mathrm{e}^{-2\gamma_\mathrm{E}}$ corresponds to the canonical choice [24].

We can now bring together real- and virtual-gluon results by referring to our generic expression for the cross-section as given by (7). First, we write

$$\mathcal{W} = \left\langle \mathrm{Tr}\left(U^\dagger(\bar{C}^{z'})U(C^z)\right)\right\rangle = U_{C,\mathrm{cusp}}U_{\bar{C},\mathrm{cusp}}U_{C\bar{C},\mathrm{cusp}}\hat{W}+\mathcal{O}\left(\frac{1}{Q^2}\right), \quad (52)$$

where the factor $\hat{W}$ contains both hard and residual soft contributions.

Then, at the cross-section level, our threshold resummation of the virtual gluons reads

$$U_\mathrm{C}U_{\bar{\mathrm{C}}} = \exp\left\{-\int_{\bar{\lambda}^2}^{Q^2}\frac{\mathrm{d}t}{t} \times \left[\ln\frac{Q^2}{t}\Gamma_\mathrm{cusp}(\alpha_\mathrm{s}(t))-\Gamma(\alpha_\mathrm{s}(t))\right]\right\}U_{\mathrm{C},0}U_{\bar{\mathrm{C}},0}. \quad (53)$$

Thus, combining the above expressions, the final result reads

$$\mathcal{W} = \exp\left\{-\int_{c/b^2}^{Q^2}\frac{\mathrm{d}t}{t} \times \left[\ln\frac{Q^2}{t}\Gamma_\mathrm{cusp}(\alpha_\mathrm{s}(t))-\Gamma(\alpha_\mathrm{s}(t))\right]\right\}\mathcal{W}_0, \quad (54)$$

with Γ_cusp and Γ given by (30) and (36), respectively, an expression obtained before in [21], employing Wilson lines as a quantity attached to quark current operators.

6 Concluding remarks

In this paper we have applied first-quantization techniques to study threshold resummation of soft gluon radiation for DY-type of processes in QCD. We have addressed our efforts in an energy regime whose lower cutoff is high enough to justify an analysis in which reference to "gluons", as dynamical degrees of freedom, continues to make sense[5].

[5] We have implicitly assumed the pre-confinement property, originally articulated in the first work of reference [25] (see also the second one), according to which the non-perturbative dynamics responsible for confinement screens color up to the infrared scale λ which sets the lower limit for the perturbative regime

First-quantization approaches to the study of relativistic quantum systems, involving either strings or world-line agents as their basic tools, have been employed by a number of authors during the last decade (the interested reader is recommended to the recent reviews [9,13]) as a viable alternative to the traditional second-quantization procedure, associated with their field theoretical casting. The main feature of this type of approach at the perturbative level is that it allows a space-time description of the field theoretical system leading to expressions that accommodate a host of Feynman diagrams at once. On the other hand, it can play a crucial role in the study of non-perturbative effects [9].

The chief issues of the present methodology can be summarized as follows.

(1) The formalism is based on the Polyakov world-line path integral, whose particular merit is that it is structured in terms of the spin factor, a quantity which is path-determined and which accounts geometrically for the spin of the propagating particle entity. Within this framework, the quark–gluon dynamics is embodied in the expectation value of open/closed Wilson lines. Thus, quantities of interest for us, such as amplitudes or cross-sections, can be solely described in terms of appropriately weighted integrals over Wilson contours.
(2) We have based our considerations on the large scale Q^2 in order to apply the mathematically well-founded fact that the *local* characteristics (like endpoints, cusps, etc.) of the contours involved in the path integral can be dissected out (cf. (52)). In this way we calculated the resummed expression for soft-gluon emission that gives rise to the Sudakov factor.
(3) We reiterate that our primary goal has not been to reproduce known results, but to show *how* to obtain them *on the cross-section level* using the world-line Polyakov path integral. It is obvious that this type of approach can be used to describe DIS-type processes as well (see [26] from which the present investigation partly derives). Furthermore, since we are not obliged to use special type of paths and we can always stay within the Euclidean formulation – at least as long as we care about the leading behavior of quantities like the cross-section, the present investigation may pave the way to extend this type of approach to the large transverse distance regime, where we shall meet power corrections signaling non-perturbative contributions [21,24,27–30]. Thus, we hope that this formalism can be extended to the calculation of the non-perturbative effects of the expectation value of Wilson loops by having recourse to the extensive existing literature [9,31] on the subject.

Appendix

Our task is to establish (19) in the text. Performing the integration entering the right hand side of (18), one obtains

$$I_3^{(\mathrm{a})} = \frac{1}{4\pi^2}\left(-\pi\mu^2\right)^{(4-D)/2}\Gamma\left(\frac{D}{2}-1\right)\frac{1}{4-D}\frac{1}{D-3}2w$$
$$\times\left\{wF\left(1,\frac{D}{2}-1;\frac{D-1}{2};1-w^2\right)\right. \tag{A.1}$$
$$\left.+\ \frac{1}{2}\left[2(1-w)\right]^{2-D/2}F\left(1,\frac{D}{2}-1;\frac{D-1}{2};\frac{1+w}{2}\right)\right\}.$$

Setting $D=4$, we obtain

$$F\left(1,1;3/2;1-w^2\right) = \frac{\gamma}{\sinh\gamma\cosh\gamma} \tag{A.2}$$

and

$$F\left(1,1;3/2;\frac{1+w}{2}\right) = \frac{\gamma}{\sinh\gamma} - \mathrm{i}\frac{\pi}{\sinh\gamma}. \tag{A.3}$$

As the imaginary part in the above expression will cancel against its counterpart in the conjugate expression, it can be dropped as far as the cross-section is concerned.

Denoting the expression inside the curly brackets on the r.h.s. of (A.1) by $f_D(w)$ and setting

$$f_D(w) = f_4(w) + (4-D)\frac{f_D(w)-f_4(w)}{4-D}, \tag{A.4}$$

we realize that the second term on the r.h.s. will lead to finite terms that depend solely on w and which will cancel against similar contributions of the same sort coming from the other terms entering (12). Putting everything together, one finally arrives at (19).

To establish the result given by (22), we first note that (21) gives

$$I_3^{(\mathrm{b})} = \frac{1}{4\pi^2}\left(-\pi\frac{\mu^2}{\lambda^2}\right)^{(4-D)/2}$$
$$\times\Gamma\left(\frac{D}{2}-1\right)\frac{1}{(4-D)^2}\left(\frac{2v\cdot v'}{|v|^2}\right)^{(4-D)/2}$$
$$\times\left[F\left(\frac{D}{2}-1,2-\frac{D}{2};3-\frac{D}{2};-\frac{2v\cdot v'}{|v|^2}\right)\right.$$
$$\left.+\left(1+\frac{2v\cdot v'}{|v|^2}\right)^{2-D/2}-\left(\frac{2v\cdot v'}{|v|^2}\right)^{2-D/2}\right]. \tag{A.5}$$

Then, in the limit $D\to 4$ one easily retrieves (22).

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